Faculty of Science, Technology, Engineering and Mathematics
MST210 Mathematical methods, models and modelling

MST210

TMA 05 2019J

Covers Units 12, 13 and 14 **Cut-off date 12 February 2020**

The module website gives details of how to submit assignments for this module. If you have a disability that makes it difficult for you to attempt any of the questions, then please contact your Student Support Team or your tutor for advice.

The marks allocated to each part of each question are indicated in brackets in the margin.

You may use Maxima to check answers, but it should not be used otherwise to answer TMA questions except where explicitly indicated. TMA answers must show full working to attract full marks.

In order to encourage you to present your solutions to the TMA questions in a good mathematical style, your tutor will comment on how you:

- use correct mathematical notation
- define any symbols that you introduce in formulating and solving a problem
- give references for standard formulas and derivations
- include comments and explanations within your mathematics
- explicitly state results and conclusions, giving answers to an appropriate degree of accuracy and interpreting answers in the context of the question
- draw diagrams and graphs.

These features are seen as being essential to complementing your mathematical skills. Your tutor will make comments on how well you achieve these objectives and give you guidance on how to satisfy them.

Five of the marks for this TMA will be allocated to the way you write your solutions. It is expected that most students will receive the majority of these presentation marks; such marks are included in TMAs to encourage, and emphasise the need for, thinking about how you present your mathematics.

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Question 1 (Unit 12) – 16 marks

Consider the pair of differential equations

$$\frac{dx}{dt} = y - 2, \quad \frac{dy}{dt} = x^2 - y^2.$$

(a) Find all the equilibrium points of these equations. [*4*]

(b) Classify each equilibrium point of this non-linear system as far as possible by considering the Jacobian matrix. [*12*]

Question 2 (Unit 12) – 10 marks

Consider the pair of differential equations

$$\frac{dx}{dt} = (x^2 - 4)(y + 2), \quad \frac{dy}{dt} = (x - 1)(y^2 - 9).$$

(a) Find all the equilibrium points of these equations. [*6*]

(b) Using Maxima, or otherwise, plot the phase portrait of these equations. [*2*]

(c) Use your phase portrait to state the start and end of the phase path that goes through the origin. [*2*]

Question 3 (Unit 12) – 12 marks

Consider the pair of differential equations

$$\frac{dx}{dt} = (x - \pi)\sin y + \sin x \cos y, \quad \frac{dy}{dt} = \sin x + \pi \cos y.$$

(a) Show that $\left(\pi, \frac{\pi}{2}\right)$ is an equilibrium point of these equations. [*3*]

(b) Classify this equilibrium point of this non-linear system as far as possible by considering the Jacobian matrix. [*9*]

Question 4 (Unit 13) – 16 marks

Consider the periodic function $f(t)$ with fundamental interval $-\pi \leq t \leq \pi$ that is defined by

$$f(t) = \begin{cases} -2t - \pi & \text{for } -\pi \leq t < 0, \\ 2t - \pi & \text{for } 0 \leq t < \pi, \end{cases}$$

$f(t + 2\pi) = f(t)$.

(a) Sketch the graph of the function f for $-3\pi \leq t \leq 3\pi$, and hence state whether the function is even, odd, or neither even nor odd. [*4*]

(b) Calculate the Fourier series for $f(t)$. [*12*]

Question 5 (Unit 13) – 8 marks

Consider the function $f(t)$ defined on the interval $0 \leq t < 2$ by

$$f(t) = t(2-t).$$

(a) Sketch the graph of the odd extension f_{odd} of f for $-6 \leq t \leq 6$, and hence state the fundamental period of the odd extension. [4]

(b) Sketch the graph of the even extension f_{even} of f for $-6 \leq t \leq 6$, and hence state the fundamental period of the even extension. [4]

Question 6 (Unit 14) – 16 marks

The temperature distribution $\Theta(x,t)$ along an insulated metal rod of length L is described by the differential equation

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{1}{D}\frac{\partial \Theta}{\partial t} \quad (0 < x < L,\ t > 0),$$

where $D \neq 0$ is a constant. The rod is held at a fixed temperature of 0° C at one end and is insulated at the other end, which gives rise to the boundary conditions $\Theta(0,t) = 0$ and $\Theta_x(L,t) = 0$, for $t > 0$.

The initial temperature distribution in the rod is given by

$$\Theta(x,0) = 0.5 \sin\left(\frac{7\pi x}{2L}\right) \quad (0 \leq x \leq L).$$

(a) Use the method of separation of variables, with $\Theta(x,t) = X(x)\,T(t)$, to show that the function $X(x)$ satisfies the differential equation

$$X'' - \mu X = 0 \tag{1}$$

for some constant μ. Write down the corresponding differential equation that $T(t)$ must satisfy. [3]

(b) Write down the boundary conditions that $X(x)$ must satisfy.

Consider the functions

$$X_n(x) = \sin(k_n x), \quad \text{where } k_n = \frac{(2n-1)\pi}{2L} \text{ and } n = 1, 2, 3, \ldots.$$

Show that function $X_n(x)$ satisfies the boundary conditions that you found. Show that $X_n(x)$ satisfies differential equation (1) for some constant μ (which you should specify). [5]

(c) Solve the differential equation found in part (a) that the function $T(t)$ must satisfy. [3]

(d) Use your answers to write down a family of product solutions $\Theta_n(x,t) = X_n(x)\,T_n(t)$ that satisfy the first two boundary conditions. Hence show that the general solution of the partial differential equation may be expressed as

$$\Theta(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{D(2n-1)^2\pi^2 t}{4L^2}\right) \sin\left(\frac{(2n-1)\pi x}{2L}\right). \tag{2}$$

(e) Find the particular solution that satisfies the given initial temperature distribution. [3]

Question 7 (Unit 14) – 17 marks

The vibration of a column of air in an organ pipe can be modelled by the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} \quad (0 < x < L,\ t > 0), \tag{2}$$

where $V > 0$ is a constant.

One end of the pipe is closed, which corresponds to the boundary condition $u(0,t) = 0$, for $t > 0$. The other end of the pipe is open, which corresponds to the boundary condition $u_x(L,t) = 0$, for $t > 0$.

Applying the method of separation of variables, with $u(x,t) = X(x)\,T(t)$, gives the boundary-value problem

$$X'' = \mu X, \quad X(0) = 0, \quad X'(L) = 0, \tag{3}$$

and the differential equation

$$\ddot{T} = \mu V^2 T. \tag{4}$$

(a) Suppose that $\mu < 0$, so $\mu = -k^2$ for some $k > 0$. Find the non-trivial solution $X(x)$ that satisfies equations (3), stating clearly what values k is allowed to take. [8]

(b) Write down the general solution of equation (4) for the case $\mu = -k^2$. [2]

(c) You may assume that if $\mu \geq 0$, then only the trivial solution satisfies equations (3). Use this assumption to write down the general solution of the partial differential equation (2) that satisfies the boundary conditions, by combining your solutions to parts (a) and (b). [2]

(d) Write down the solution that corresponds to the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 2\sin\left(\frac{5}{2}\frac{\pi x}{L}\right)$. [5]